Steger code, which he developed, and to Dr. Jim Jacocks for plotting the streak-line patterns of Fig. 3. The support of Arvin-Calspan to the author during this work is also gratefully acknowledged.

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Reynolds Stresses for Unsteady Turbulent Flows

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Introduction

REYNOLDS-averaged equations are currently employed in the numerical analysis of turbulent flows. In these equations an "apparent stress" that requires empirical data (e.g., eddy viscosity) is introduced to close the system and permit calculation of the flowfield. A predicament arises when calculating *unsteady* turbulent flows in which the empirical apparent stress level is of concern. For example, if all of the apparent stress obtained experimentally over the whole frequency spectrum were used in the calculation, this value would be added to the stress generated by the numerical algorithm, causing overestimation of the turbulence. This error would produce excessive dissipation of the unsteady phenomenon with the potential attendant disappearance of the unsteady solution. It is the purpose of this Note to address the procedure required to evaluate the frequency range of the Reynolds stress to be used for unsteady turbulent flows.

Reynolds-Averaged Equations

To demonstrate the procedure, the Reynolds-averaged equations will be examined for two-dimensional, incompressible flow. Only the x-momentum equation needs to be examined for this purpose,

$$(\rho u)_t + (\rho u^2 - \sigma_{11})_x + (\rho uv - \tau)_y = 0 \tag{1}$$

where $\sigma_{11} = -p + 2\mu u_x$ and $\tau = \mu(u_y + v_x)$.

To define the mean and fluctuating quantities in the classical manner, let

$$u = \bar{u}(x,y,t) + u'(x,y,t) \tag{2}$$

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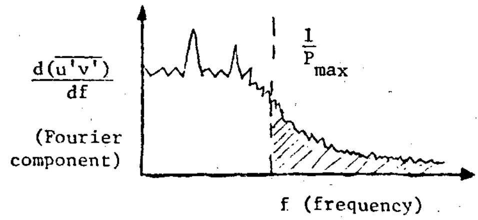

Fig. 1 Typical power spectral density plot.

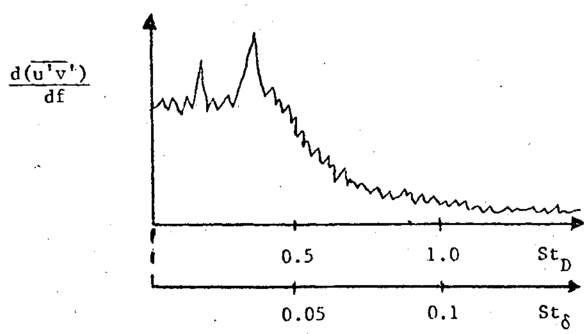

Fig. 2 Power spectral density vs Strouhal number.

$$v = \bar{v}(x,y,t) + v'(x,y,t) \tag{3}$$

$$p = \bar{p}(x,y,t) + p'(x,y,t) \tag{4}$$

where the fluctuating terms vanish over some time interval P, i.e.,

$$\frac{1}{P}\int_t^{t+P} u'\,\mathrm{d}t = 0 \tag{5}$$

The value of this P interval (as yet undefined) will be shown to be extremely important to the problem at hand. Inserting these values for the flow parameters into the x-momentum equation and integrating over the time interval P produces

$$(\rho\bar{u})_t + \left[\rho\bar{u}\bar{v} - \bar{\tau} + \frac{\rho}{P}\int_t^{t+P} u'v'\,\mathrm{d}t\right]_y + \left[\rho\bar{u}^2 - \bar{\sigma}_{11} + \frac{\rho}{P}\int_t^{t+P} u'^2\,\mathrm{d}t\right]_x = 0 \tag{6}$$

The two integrals are called the "apparent stress" terms. The second integral has little importance in most engineering problems since it is small compared to σ_{11}. However, the first integral is usually larger than the shear stress term τ and must be analyzed carefully. Defining

$$\overline{\overline{u'v'}} = \frac{1}{P}\int_t^{t+P} u'v'\,\mathrm{d}t \tag{7}$$

the Reynolds stress is therefore

$$\tau_t = -\rho\overline{\overline{u'v'}} \tag{8}$$

This average is taken over an interval P.

In the numerical calculation of unsteady turbulent flow all disturbances are resolved to within $\Delta s(\sqrt{\Delta x^2 + \Delta y^2})$ in space and Δt in time. No disturbance waves inside these increments can be resolved. This statement explains how the Reynolds-averaged equations are utilized, that is, the interval P over which the integration is taken for the averaging process must be related to a time step. Thus, $P = \Delta t$. Note that consideration of wave propagation can also produce one other relationship, $P=\Delta s/U$, where U is the freestream velocity.

In practice one should use the *maximum* value of the two periods. In most engineering problems $\Delta s/U$ will be maximum. This restriction limits the resolution of numerical frequencies to those below a value of $f_{\max}=1/P_{\max}$.

In summary, the use of the Reynolds-averaged equations for numerically solving time-dependent flows implies an averaging over an interval P that is related to the step size of the numerical grid.

Empirical Reynolds Stress

With this strategy, requirements can be imposed upon the experimental determination of Reynolds' stresses. Measured values of $u'v'$ are required to model turbulence. This can be accomplished (admittedly with great difficulty) by use of a hot-wire anemometer or laser Doppler velocimeter. When two wire hot-film anemometers are used, Reynolds stress components can be obtained by using the ac voltage output, which corresponds to mass flux per unit area, from the two wires. For example,

$$\overline{u'v'}=k(\overline{e_1^2}-\overline{e_2^2}) \tag{9}$$

where e_1 and e_2 are the ac voltage outputs from the two wires and the constant $k=0.5/\rho^2$. From these data a spectral analysis of $\overline{u'v'}$ can be performed and the Fourier components ascertained for a wide range of frequencies. By recording $u'v'$ vs time and employing a harmonic analyzer the important Fourier components may be extracted.

A typical power spectral density plot of these coefficients is shown in Fig. 1 where the usual definition is

$$\overline{u'v'}=\int_0^\infty \frac{\mathrm{d}(\overline{u'v'})}{\mathrm{d}f}\,\mathrm{d}f \tag{10}$$

For the numerical analysis only the Reynolds' stresses for the frequency range above $f=1/P_{\max}$ are of interest, since those frequencies below $1/P_{\max}$ are captured by the numerical solution. Hence $\overline{u'v'}$ is redefined for numerical calculation as

$$\overline{\overline{u'v'}}=\int_{1/P}^\infty \frac{\mathrm{d}(\overline{u'v'})}{\mathrm{d}f}\,\mathrm{d}f \tag{11}$$

This integral represents only the higher end of the frequency spectrum shown in Fig. 1. The lower end of the spectrum will be computed using the complete Navier-Stokes equations.

A very important parameter used for unsteady turbulent flows is the Strouhal number defined as $St_A=fA/U$ where f is the frequency of the disturbance, U the freestream or uniform flow velocity, and A a dimension that, for bodies of revolution, is usually the body diameter D or, for unsteady flows, can be the shear layer thickness δ. The power spectral density for the Reynolds stresses can be plotted vs the Strouhal number, as shown in Fig. 2.

From previous experimental results, it is known that most of the turbulent energy is concentrated in the Strouhal number range $St_D=0\text{-}0.6$ for most configurations. In the wake of a two-dimensional circular cylinder, the maximum Strouhal number St_D recorded is 0.21. For ogive cylinders at high angles of attack most of the energy in the wake is concentrated at $0.2\leq St_D\leq 0.6$ for freestream Mach numbers up to 3 and cross-flow Mach numbers up to 2.[1] Self-excited shock oscillations on spike-tipped bodies of revolution placed at zero angle of attack in a supersonic flow at Mach 3 generate a Strouhal number St_D of about 0.2 (Ref. 2) and of about 0.22 for the experimental configuration of Ref. 3. Self-excited wave oscillations generated in water on similar two-dimensional spike-tipped bodies have a Strouhal number based on spike length $St_L=0.16$, while oscillations generated in air and water over rectangular cavities have $St_L=0.3$.[4]

If a Strouhal number based on shear-layer thickness St_δ were used as a parameter, most of the turbulent energy would be concentrated in a Strouhal number range one order of magnitude smaller than for St_D. For example, for the spike-tipped body experiment of Ref. 3, St_D is equal to 0.22 while St_δ is equal to 0.02 for maximum oscillations. The shock oscillations subside for $St_\delta=0.04$. Thus, most turbulent flows must have a stress distribution where the energy is concentrated at $St_\delta<1.0$. Then, if $St_\delta \ll 1.0$, this condition is analogous to a state of "frozen" turbulence and the total value of the eddy viscosity should be used in the Reynolds-averaged equations. Conversely, a condition where $St_\delta \gg 1$ is analogous to "equilibrium" turbulence and the eddy viscosity value used should be zero.

Conclusion

To analyze unsteady turbulent flows, the Reynolds-averaged equations are used, *but* averaged over a period P which is related to the grid size,

$$P=\max \text{ of } \Delta t,\ \Delta s/U$$

Therefore the Reynolds' stress obtained from experiment and used in the numerical calculation is only that portion of the $u'v'$ frequency spectrum above the frequency $=1/P$,

$$\overline{\overline{u'v'}}=\int_{1/P}^\infty \frac{\mathrm{d}\overline{u'v'}}{\mathrm{d}f}\,\mathrm{d}f$$

Depending on the value of the Strouhal number based on the shear-layer thickness, the total experimental value of the eddy viscosity can be used in the equations if $St_\delta \ll 1$ and the eddy viscosity can be set equal to zero if $St_\delta \gg 1$. If $St_\delta=0(1)$, the correct frequency spectrum of the experimental value of the eddy viscosity must be used.

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Buckling of Sinusoidally Corrugated Plates under Axial Compression

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Introduction

WIDELY used as a lightweight structure in a number of applications, the corrugated plate has usually been

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